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Bounded Perturbations of Controllable Systems. II

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1. INTRODUCTION

In a recent paper [1] the author showed that if the control system

$$\dot{x} = A(t)\,x + k(t, u) \tag{1}$$

is completely controllable and if the function $g(t, x, u)$ is bounded, then the perturbed system

$$\dot{x} = A(t)\,x + k(t, u) + g(t, x, u) \tag{2}$$

is completely controllable provided a certain convexity condition is satisfied by the function $k + g$. Similar results have been obtained when the function $k(t, u)$ is linear in u by Hermes [2], Lukes [3], and Aronsson [4, 5] without the convexity condition.

The purpose of this paper is to eliminate the convexity condition of the author's earlier work. This goal was partially accomplished in the earlier paper [1] where it was shown that if g satisfies a global Lipschitz condition in x and is bounded, then system (2) is approximately controllable; i.e., any initial point can be steered arbitrarily close to the desired target. This type of controllability is, of course, sufficient for a number of applications.

In Section 2 of this paper we use an approach based on that of Aronsson to show that if g satisfies local Lipschitz conditions in x and in u and is (integrably) bounded, then system (2) is completely controllable. In fact, we obtain somewhat more and will generalize several of Aronsson's [4, 5] results, as well as the related result of Hermes [2], to perturbations of system (1). Our results do not completely generalize the earlier work of the author [1]. However, we do replace the convexity condition on the function $k + g$ by rather mild local Lipschitz conditions on g. Also, in this paper we require little more than Carathéodory-type conditions on the system, rather than continuity. This is a desirable generalization for many systems. Further, in Section 3 we show that our results can be extended to the problem of approximate controllability using continuous controllers.

2. Main Results

Take $x \in E^n$, $u \in E^m$ and assume the matrix function A is Lebesque integrable on the compact interval $I = [t_0, T]$. Suppose the vector function k is measurable in t for fixed u and continuous in u uniformly in t on compact subsets of E^m (e.g. if k is continuous in (t, u)). The class of admissible control functions is the set of all bounded, measurable functions $u: I \to E^m$. We assume that if u is an admissible control function then $k(\cdot, u(\cdot))$ is integrable on I. The class of admissible perturbations is the set $\mathcal{O}\!\ell$ of functions $g: I \times E^n \times E^m \to E^n$ satisfying the following four conditions.

(1) g is measurable in t for fixed (x, u) and continuous in (x, u) for fixed t.

(2) $g(t, o, o)$ is integrable on I.

(3) For each $M > 0$, there is an integrable function $a(t)$ on I such that if $|u| \leqslant M$, then

$$|g(t, x_1, u) - g(t, x_2, u)| \leqslant a(t) \, | x_1 - x_2 | \, .$$

(4) For each $M > 0$, there is an integrable function $b(t)$ on I and a continuous nondecreasing function $\eta(s)$, satisfying $\eta(0) = 0$, such that if $| u_1 | \leqslant M$ and $| u_2 | \leqslant M$, then

$$|g(t, x, u_1) - g(t, x, u_2)| \leqslant b(t) \, (| x | + 1) \, \eta(| u_1 - u_2 |).$$

We say that system (2) is *completely controllable* if given any pair x_0, $x_1 \in E^n$ there is an admissible control function u such that the solution of

$$\dot{x} = A(t) \, x + k(t, u(t)) + g(t, x, u(t)), \qquad x(t_0) = x_0 \, ,$$

satisfies $x(T) = x_1$. Clearly system (1) is a special case of system (2) with the admissible perturbation $g(t, x, u) \equiv 0$.

The following notation and preliminary results will be used throughout the paper. Let $X(t)$ denote the fundamental matrix solution of $\dot{x} = A(t) \, x$, with $X(t_0)$ the identity matrix. If u is an admissible control function, then the corresponding solution of system (1) with initial condition $x(t_0) = x_0$ is given by

$$x(t) = X(t) \, x_0 + X(t) \int_{t_0}^{t} X^{-1}(s) \, k(s, u(s)) \, ds.$$

Since we are interested in the collection of points $x(T)$ corresponding to the various admissible control functions u, we let

$$\phi(u) = X(T) \int_{t_0}^{T} X^{-1}(s) \, k(s, u(s)) \, ds$$

and take $\Phi_0(u) = X(T)\,x_0 + \phi(u)$ when the initial point is specified. Similarly, the variation of constants formula for system (2) motivates considering

$$\Phi(u) = X(T)\,x_0 + \phi(u) + X(T) \int_{t_0}^{T} X^{-1}(s)\, g(s, x(s; u), u(s))\, ds$$

where $x(t; u)$ denotes the solution of system (2) corresponding to the control function u and satisfying the initial condition $x(t_0) = x_0$.

For $\rho > 0$ let $S(\rho)$ denote the closed ball in E^n or E^m with radius ρ centered at the origin. A necessary and sufficient condition for system (1) to be completely controllable [1, Prop. 1] is that for every $r > 0$ there exists $\rho > 0$ such that

$$\int_{t_0}^{T} X^{-1}(s)\, k(s, S(\rho))\, ds \supseteq S(r). \tag{3}$$

Note that if we let

$$\Omega(\rho) = \{\text{admissible controls } u\text{: } u(t) \in S(\rho) \text{ for } t \in I\},$$

then condition (3) is equivalent to

$$\phi[\Omega(\bar{\rho})] \supseteq S(r)$$

for some $\bar{\rho} > 0$. Hence system (1) is completely controllable if and only if

$$\bigcup_{i=1}^{\infty} \phi[\Omega(\rho_i)] = E^n$$

for any unbounded sequence of positive numbers $\{\rho_i\}$. Finally, if system (1) is completely controllable and if $r > 0$, then there exists $\rho_r > 0$ such that

$$\Phi_0[\Omega(\rho_r)] \supseteq S(r).$$

The following theorem contains the main result of this paper. We say that a function $g(t, x, u)$ is *integrably bounded* if there exists an integrable function $\alpha(t)$ on I such that

$$|\, g(t, x, u)| \leqslant \alpha(t)$$

for all $(t, x, u) \in I \times E^n \times E^m$. Since I is compact, all bounded functions are clearly integrably bounded.

THEOREM 1. *Suppose system* (1) *is completely controllable and the perturbation $g \in \mathcal{O}\!\mathcal{l}$ is integrably bounded, then system* (2) *is completely controllable.*

Proof. Let the initial point x_0 be fixed and let $r > 0$ be given. We will show that there exists $\rho > 0$ satisfying

$$\Phi[\Omega(\rho)] \supseteq S(r).$$

Since r is arbitrary, it follows that system (2) is completely controllable. The integrable bound on the function g implies there exists a finite number d satisfying

$$|\Phi(u) - \Phi_0(u)| < d$$

for all admissible control functions u. So choose $\rho > 0$ such that

$$\Phi_0[\Omega(\rho)] \supseteq S(r + d + 1).$$

We now show that

$$\Phi[\Omega(\rho)] \supseteq S(r).$$

Following Aronsson [5] we partition E^n into cubes defined by

$$k_i/2^p \leqslant x_i \leqslant (k_i + 1)/2^p, \qquad i = 1, 2,..., n,$$

where x_i is a component of $x \in E^n$, $\{k_i\}$ are arbitrary integers, and p is a natural number. Let E_0 be the union of all such cubes that are contained in $S(r + d + 1)$, where p is fixed sufficiently large that $E_0 \supseteq S(r + d)$. Let $X_1, X_2,..., X_N$ be the vertices of the cubes in E_0. Choose continuous functions $\mu_1(x), \mu_2(x),..., \mu_N(x)$ satisfying:

$$\text{(a)} \quad 0 \leqslant \mu_j(x) \leqslant 1; \qquad \sum_{j=1}^{N} \mu_j(x) = 1 \qquad \text{if} \qquad x \in E_0,$$

$$\text{(b)} \quad x = \sum_{j=1}^{N} \mu_j(x) X_j \qquad \text{if} \qquad x \in E_0,$$

(see [5, Proof of Lemma 1]). Now $E_0 \subseteq S(r + d + 1) \subseteq \Phi_0[\Omega(\rho)]$ and so there exist controllers $u_j \in \Omega(\rho)$ such that

$$\Phi_0(u_j) = X_j, \qquad j = 1, 2,..., N.$$

For N-vectors $\mu = (\mu_1,..., \mu_N)$ satisfying property (a) above there exists a continuous family of N-partitions of I, $\mu \mapsto \{A_1(\mu),..., A_N(\mu)\}$, such that the integrable function

$$h(t, \mu) = \begin{cases} X(T)\, X^{-1}(t)\, k(t, u_1(t)) & \text{for } t \in A_1(\mu), \\ X(T)\, X^{-1}(t)\, k(t, u_2(t)) & \text{for } t \in A_2(\mu), \\ \vdots & \\ X(T)\, X^{-1}(t)\, k(t, u_N(t)) & \text{for } t \in A_N(\mu), \end{cases}$$

satisfies

$$\int_{t_0}^{T} h(s, \mu)\, ds = \sum_{j=1}^{N} \mu_j \phi(u_j)$$

for all such μ [6, pp. 372–373]. The sets $A_1(\mu),\ldots, A_N(\mu)$ are disjoint, measurable and $\bigcup_{j=1}^{N} A_j(\mu) = I$. Define

$$u(t, \mu) = \begin{cases} u_1(t) & \text{for } t \in A_1(\mu), \\ u_2(t) & \text{for } t \in A_2(\mu), \\ \vdots & \\ u_N(t) & \text{for } t \in A_N(\mu). \end{cases}$$

Then $u(t, \mu) \in \Omega(\rho)$ and we have

$$X(T)\, X^{-1}(t)\, k(t, u(t, \mu)) = h(t, \mu) \qquad \text{for } t \in I.$$

Hence

$$\phi(u(\cdot, \mu)) = \sum_{j=1}^{N} \mu_j \phi(u_j)$$

for all such μ; which implies that

$$\Phi_0(u(\cdot, \mu)) = \sum_{j=1}^{N} \mu_j \Phi_0(u_j) = \sum_{j=1}^{N} \mu_j X_j\,.$$

Denoting $u_x(t) \equiv u(t, \mu(x))$ gives

$$x = \sum_{j=1}^{N} \mu_j(x)\, X_j = \Phi_0(u_x)$$

for all $x \in E$. Define a map $T: E_0 \to E^n$ by

$$T(x) = \Phi(u_x).$$

Using $g \in \mathcal{O}\!\mathcal{l}$, Aronsson [5, Proof of Lemma 1] shows that the map

$$x \mapsto X(T) \int_{t_0}^{T} X^{-1}(s)\, g(s, x(s; u_x), u_x(s))\, ds$$

is continuous. Since u_x is continuous in x in the L^∞-norm, the uniform continuity of k in u implies that the map $x \mapsto \phi(u_x)$ is continuous on E_0. Hence T is continuous. Since

$$|\, x - T(x)| = |\, \Phi_0(u_x) - \Phi(u_x)| < d$$

for all $x \in \partial S(r + d)$ (in fact, for all x), we have

$$T[S(r + d)] \supseteq S(r)$$

using a basic topological covering theorem [6, p. 251]. Therefore, $\Phi[\Omega(\rho)] \supseteq S(r)$ and the result is proved.

Remarks. It is obvious from the earlier work of the author [1] and that of Aronsson [4] that numerous examples can be constructed. Therefore we will omit any examples from this paper.

The results on perturbations of linear systems done by Lukes [3] and the mild local Lipschitz conditions on the functions in $\mathcal{O}\!\mathcal{l}$ lead the author to believe that the results of Theorem 1 are valid without the assumption that $g \in \mathcal{O}\!\mathcal{l}$. However this conjecture remains an open question.

It should be noted that if $k(t, \alpha u) = \alpha k(t, u)$ for each real number α, $t \in I$, $u \in E^m$, then all of Aronsson's results [4, 5] remain valid for system (2). To see this replace Aumann's theorem by a more general result of Castaing [7, Theorem 7.0] (see also [8, 9]), note that

$$\Phi_0[\Omega(\alpha\rho)] = \alpha\Phi_0[\Omega(\rho)],$$

and apply the modifications used in the proof of Theorem 1 to the arguments used by Aronsson [4, 5]. From this it is clear that if one is able to specify the relationship between $\Phi_0[\Omega(\alpha\rho)]$ and $\alpha\Phi_0[\Omega(\rho)]$ for a system (1), then more general results, like those of Aronsson, can be obtained for system (2).

It is easy to see from our proof of Theorem 1 and that of Aronsson [4, Sect. 4] that we can extend our proof to obtain the following result. This is a slight extension of Theorem 1 (see [4, Examples 1–3]).

THEOREM 2. *Suppose that for each initial point $x_0 \in E^n$ there exists a sequence $\{E_i(t)\}$ of nonempty, compact set-valued functions which are continuous on I in the Hausdorff metric and an unbounded positive sequence $\{r_i\}$ such that*

$$co k(t, E_i(t)) \supseteq k(t, S(r_i))$$

for all $t \in I$. If system (1) is completely controllable and there is a number ρ such that the perturbation $g \in \mathcal{O}\!\mathcal{l}$ satisfies

$$| \Phi(u) - \Phi_0(u)| \leqslant \rho$$

for $u \in \Omega(E_i)$, $i = 1, 2,\ldots$, then system (2) is completely controllable.

3. Continuous Controllers

In this section we will consider an extension of the previous results to the situation where the class of admissible controllers is limited to continuous control functions. For this type of restriction it is necessary to assume that the functions k and g are continuous.

We say that system (2) is *approximately controllable with continuous controls* if given any pair x_0, $x_1 \in E^n$ and any $\epsilon > 0$ there is a continuous control function u such that the solution of

$$\dot{x} = A(t)\,x + k(t, u(t)) + g(t, x, u(t)), \qquad x(t_0) = x_0,$$

satisfies $|x(T) - x_1| < \epsilon$. This type of controllability is of interest in a number of physical applications.

THEOREM 3. *Suppose system* (1) *is completely controllable and the perturbation* $g \in \mathcal{C}l$ *is integrably bounded, then system* (2) *is approximately controllable with continuous controls.*

Proof. Let x_0, $x_1 \in E^n$ and $\epsilon > 0$ be given. By Theorem 1 there exists a measurable control function v which is bounded, say by M, on I such that the corresponding solution $x(\cdot; v)$ of (2) satisfies $x(T; v) = x_1$. Let a be the Lipschitz constant for the continuous function g guaranteed by condition 3 for $|u| \leqslant M$, and take

$$d = \max\{|X(t)\,X^{-1}(s)|: t, s \in I\},$$
$$N = 8d \exp(da(T - t_0)).$$

Let $\alpha(\cdot)$ be the integral bound on g, there exists [10, p. 176] a $\delta_1 > 0$ such that if E is a measurable subset of I with measure less than δ_1, then

$$\int_E \alpha(s)\,ds < \epsilon/N.$$

Further, k is continuous and therefore bounded on the compact set

$$\{(t, u): t \in I, |u| \leqslant M\},$$

say by b. Choose δ, with $\delta < \delta_1$, such that if E is a measurable subset of I and the measure of E is less than δ, we have

$$\int_E b\,ds < \epsilon/N.$$

By Luzin's theorem [10, p. 159], there exists a continuous function u such that the measure of the set

$$E = \{t \in I: u(t) \neq v(t)\}$$

is less than δ and

$$|u(t)| \leqslant |v(t)| \leqslant M \qquad \text{for } t \in I.$$

Then the solution $x(\cdot; u)$ of (2) corresponding to the continuous controller u satisfies, for $t \in I$,

$$
\begin{aligned}
| x(t; u) - x(t; v)| &\leqslant d \int_{t_0}^{t} | k(s, u(s)) - k(s, v(s))| \, ds \\
&\quad + d \int_{t_0}^{t} | g(s, x(s; u), u(s)) - g(s, x(s; u), v(s))| \, ds \\
&\quad + d \int_{t_0}^{t} | g(s, x(s; u), v(s)) - g(s, x(s; v), v(s))| \, ds \\
&\leqslant 2d \int_E b \, ds + 2d \int_E \alpha(s) \, ds + da \int_{t_0}^{t} | x(s; u) - x(s; v)| \, ds \\
&\leqslant 4d\epsilon/N + da \int_{t_0}^{t} | x(s; u) - x(s; v)| \, ds.
\end{aligned}
$$

Hence

$$| x(t; u) - x(t; v)| \leqslant 4d\epsilon N^{-1} \exp(da(T - t_0)),$$

by Grönwall's inequality. Therefore,

$$| x(T; u) - x_1 | = | x(T; u) - x(T; v)| \leqslant \epsilon/2.$$

This concludes the proof.

The following immediate corollary of Theorem 3 is of interest in its own right.

COROLLARY. *A necessary and sufficient condition for system* (1) *to be completely controllable is that it be approximately controllable with continuous controls.*

Proof. The necessary part follows from Theorem 3 with $g \equiv 0$. The sufficiency follows from [1, Prop. 1] after noting that the set of attainability [6, p. 69] of system (1) is convex when using measurable controllers [8, Theor. 1].

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